

GEOMETRY OF MEAN VALUE SETS FOR GENERAL DIVERGENCE FORM UNIFORMLY ELLIPTIC OPERATORS

ASHOK ARYAL AND IVAN BLANK

ABSTRACT. In the Fermi Lectures on the obstacle problem in 1998, Caffarelli gave a proof of the mean value theorem which extends to general divergence form uniformly elliptic operators. In the general setting, the result shows that for any such operator L and at any point x_0 in the domain, there exists a nested family of sets $\{D_r(x_0)\}$ where the average over any of those sets is related to the value of the function at x_0 . Although it is known that the $\{D_r(x_0)\}$ are nested and are comparable to balls in the sense that there exists c, C depending only on L such that $B_{cr}(x_0) \subset D_r(x_0) \subset B_{Cr}(x_0)$ for all $r > 0$ and x_0 in the domain, otherwise their geometric and topological properties are largely unknown. In this paper we begin the study of these topics and we prove a few results about the geometry of these sets and give a couple of applications of the theorems.

1. INTRODUCTION

Based on the great importance of the mean value theorem in understanding harmonic functions, it is clear that analogues for operators other than the Laplacian are automatically of interest. In 1963, Littman Stampacchia, and Weinberger showed that if μ is a nonnegative measure on Ω and u is the solution to

$$(1.1) \quad \begin{aligned} Lu &= \mu & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{aligned}$$

and $G(x, y)$ is the Green's function for L on Ω then $u(y)$ is equal to

$$(1.2) \quad \lim_{a \rightarrow \infty} \frac{1}{2a} \int_{a \leq G \leq 3a} u(x) a^{ij}(x) D_{x_i} G(x, y) D_{x_j} G(x, y) dx$$

almost everywhere, and this limit is nondecreasing [11, Equation 8.3]. On the other hand, this formula is not as nice as the basic mean value formulas for Laplace's equation for a number of reasons. First, it is an average with weights, and not merely a simple average. Indeed, the weights in question are not even easy to estimate. Second, it is not an average over a ball or something which is even homeomorphic to a ball, but rather an average over level sets of the Green's function which do not include the central point being estimated.

The following simpler mean value theorem was stated by Caffarelli in [6, 7] and proved carefully by the second author and Hao within [3].

Theorem 1.1 (Mean Value Theorem for Divergence Form Elliptic PDE). *Let L be any divergence form elliptic operator with ellipticity λ, Λ . For any $x_0 \in \Omega$, there exists an increasing family $D_R(x_0)$ which satisfies the following:*

- (1) $B_{cR}(x_0) \subset D_R(x_0) \subset B_{CR}(x_0)$, with c, C depending only on n, λ and Λ .
- (2) For any v satisfying $Lv \geq 0$ and $R < S$, we have

$$(1.3) \quad v(x_0) \leq \frac{1}{|D_R(x_0)|} \int_{D_R(x_0)} v \leq \frac{1}{|D_S(x_0)|} \int_{D_S(x_0)} v.$$

Finally, the sets $D_R(x_0)$ are noncontact sets of the following obstacle problem:

$u \leq G(\cdot, x_0)$ such that

$$(1.4) \quad \begin{aligned} L(u) &= -\chi_{\{u < G\}} R^{-n} & \text{in } B_M(x_0) \\ u &= G(\cdot, x_0) & \text{on } \partial B_M(x_0) \end{aligned}$$

where $B_M(x_0) \subset \mathbb{R}^n$ and $M > 0$ is sufficiently large.

Although this theorem has already been shown to be useful (see for example [8] as one place where it has already been applied in this form), it is clear that the more that is known about the $D_R(x_0)$ the more useful the theorem is. It is also clear that although the fact that $B_{cR}(x_0) \subset D_R(x_0) \subset B_{CR}(x_0)$ for all R gives us some information about these sets, there is still much more that is unknown.

The present work actually originated as an attempt to better understand the solutions of a free boundary problem of Bernoulli type. In the celebrated paper of Alt and Caffarelli in 1981, nonnegative local minimizers of the functional

$$(1.5) \quad J(u) := \int_D (|\nabla u|^2 + \chi_{\{u > 0\}} Q^2)$$

are studied [1]. They are shown to exist and satisfy certain Lipschitz regularity estimates, and they obey a linear nondegeneracy statement along their free boundary. From there, Alt and Caffarelli turn to a study of the free boundary. This problem is also found (with $Q \equiv 1$) near the beginning of the text by Caffarelli and Salsa [5, Chapter 1], and the first author of this paper was working on a generalization of that problem for his dissertation. In particular, we were considering the functional

$$(1.6) \quad J_a(u) := \int_D (a^{ij} D_i u D_j u + \chi_{\{u > 0\}})$$

with uniformly elliptic a^{ij} , and that will certainly color some aspects of the current work. Unfortunately, after we started our project we learned of very nice and very recent work of dos Prazeres and Teixeira which solved some of the problems that we had intended to publish [9]. Nevertheless, their work had nothing to do with the MVT, and so we can now describe the dual purpose of the current work: First, we wish to state some theorems related to the geometry of the $D_r(x_0)$. Second, we wish to show two applications in particular which illustrate both the usefulness of the MVT, and the usefulness of our own results which give a more detailed view of properties of the $D_r(x_0)$.

The two biggest contributions that we make within this work regarding the properties of the $D_r(x_0)$ appear to be the following:

Lemma 1.2 (Density Result). *Assume $y_0 \in \partial D_r(x_0)$, and assume that c and C are the constants given in Theorem 2.2. Fix $h \in (0, 1/2)$. There exists a positive constant τ such that*

$$(1.7) \quad \frac{|B_{chr}(y_0) \cap D_r(x_0)|}{|B_{chr}(y_0)|} \geq \tau .$$

This result prevents the $D_r(x_0)$ from having what might be described as an “outward pointing cusp.”

Lemma 1.3 (Continuous Expansion). *Fix $x_0, y_0 \in \Omega$ and assume that there exists $0 < s < t$ so that y_0 is not contained in $D_s(x_0)$, and is compactly contained within $D_t(x_0)$. Then there exists a unique $r \in (s, t)$ such that $y_0 \in \partial D_r(x_0)$.*

This result allows us to state that the boundary of the mean value sets will move in a continuous fashion.

We were able to use the mean value theorem above in order to prove positive density of the contact set along the free boundary. Originally, we needed our two lemmas just mentioned in order to prove a nondegeneracy lemma for the Bernoulli problem above. Very recently, in joint work with Benson and LeCrone, the second author has extended many of the results within this work to Riemannian manifolds [2] in the case where L is the Laplace-Beltrami operator. Indeed, all of the results from Section 2 can be extended to this case, and when dealing with the obstacle problem on a compact Riemannian manifold \mathcal{M} with boundary, in order to be sure that the $D_r(x_0)$ can be extended until an r_0 where $\partial D_{r_0}(x_0)$ collides with $\partial \mathcal{M}$, we need the analogue of Lemma 1.3. (See in particular [2, Corollary 4.9].)

2. SOLID MVT FOR DIVERGENCE FORM ELLIPTIC OPERATORS

Let Ω be an open connected set in \mathbb{R}^n , and let $A(x) = (a^{ij}(x))$ be a symmetric uniformly elliptic matrix. That is for each $x \in \Omega$ we have unique matrix $a^{ij}(x)$ satisfying:

$$(2.1) \quad a^{ij} \equiv a^{ji} \quad (\text{i.e. symmetry})$$

and there exist $0 < \lambda \leq \mu < \infty$ such that

$$(2.2) \quad 0 < \lambda |\xi|^2 \leq a^{ij}(x) \xi_i \xi_j \leq \mu |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \text{ and } x \in \Omega,$$

which is called uniform ellipticity in this setting. Although there are certainly very interesting operators which are not uniformly elliptic, we will content ourselves to assume uniform ellipticity throughout this entire work.

Remark 2.1 (Analyst's Convention). Notice that with our definition we can have $L = \Delta$, but we won't have $L = -\Delta$.

We consider the divergence form operator $L := \operatorname{div}(A(x)\nabla(u))$. For any $f \in L^2(\Omega)$, we will say that u is a subsolution of $Lu = f$ (or more simply $Lu \geq f$), whenever $u \in W^{1,2}(\Omega)$ and for every $\phi \in W_0^{1,2}(\Omega)$, $\phi \geq 0$, we have

$$(2.3) \quad - \int_{\Omega} a^{ij} D_i u D_j \phi \geq \int_{\Omega} f \phi.$$

Of course, supersolutions are defined in the same way, but with the inequality in Equation (2.3) reversed.

We recall here the main MVT that is the focus of our attention:

Theorem 2.2 (MVT for divergence form elliptic PDE). *Let L be a divergence form elliptic operator as described above. For any $x_0 \in \Omega$, there exist an increasing family $D_R(x_0)$ which satisfies the following:*

- (1) *There exists c and C depending only on n, λ , and μ , such that for all $R > 0$ such that $B_{cR}(x_0) \subset \Omega$ we have $B_{cR}(x_0) \subset D_R(x_0) \subset B_{CR}(x_0)$.*
- (2) *For any v satisfying $Lv \geq 0$ in Ω and any $0 < R < S$, we have*

$$(2.4) \quad v(x_0) \leq \frac{1}{|D_R(x_0)|} \int_{D_R(x_0)} v \leq \frac{1}{|D_S(x_0)|} \int_{D_S(x_0)} v.$$

*Finally, the sets $D_R(x_0)$ are noncontact sets of the following obstacle problem:
 $u \leq G(\cdot, x_0)$ such that*

$$(2.5) \quad \begin{aligned} L(u) &= -\chi_{\{u < G\}} R^{-n} & \text{in } B_M(x_0) \\ u &= G(\cdot, x_0) & \text{on } \partial B_M(x_0) \end{aligned}$$

where $B_M(x_0) \subset \mathbb{R}^n$ and $M > 0$ is sufficiently large.

Remark 2.3 (Dependencies). It is shown in [3] that for any $R > 0$, the solution of the obstacle problem above becomes independent of the choice of M as long as it is sufficiently large, and we will always assume that that is the case. (It will be identically equal to the Green's function outside of the compact set $D_R(x_0)$.) We will frequently want to stress the dependence of the solution on R , and so, accordingly, we will refer to it as " u_R ." We will also use " $w_R := G - u_R$ " when we wish to look at a function which, at least away from x_0 satisfies the usual equations obeyed by the height function for an obstacle problem.

Remark 2.4 (Technicality). Technically, we cannot use the function $G(x, x_0)$ as boundary values in the sense of having a difference in $W_0^{1,2}$ until we suitably remove the singularity at x_0 , so within [3] they use a function that they call G_{sm} which agrees with G within a neighborhood of the boundary but which has no singularity in order to bypass this difficulty.

The function u_R is also the minimizer of

$$(2.6) \quad J_R(u, \Omega) := \int_{\Omega} (a^{ij} D_i u D_j u - 2R^{-n} u) dx$$

among functions less than or equal to G with boundary values equal to G . Note that the Green's function G of the general divergence form elliptic operator L is the analogue of the classical obstacle and u_R is that of the membrane, and here the obstacle constrains the membrane from above.

Although, as Caffarelli observed, the sets $D_R(x_0)$ are nested and comparable to balls in the sense that:

$$B_{cR}(x_0) \subset D_R(x_0) \subset B_{CR}(x_0) ,$$

we know very little about the topology of the sets. As a first small step in this direction we offer the following lemma:

Lemma 2.5 (Structure of $D_R(x_0)$). *For any $x_0 \in \Omega$ and for any $R > 0$ such that $B_{cR}(x_0) \subset \Omega$, the set $D_R(x_0)$ has exactly one component and it contains x_0 .*

Proof. Since $x_0 \in B_{cR}(x_0) \subset D_R(x_0)$, it is immediate that $x_0 \in D_r(x_0)$. Although this statement is certainly trivial, we include it because of the observation that the MVT given by Littman, Stampacchia, and Weinberger does not have this property. (See (1.2) above.)

Now for the next part, without loss of generality we can assume $x_0 = 0$. Assume for the sake of obtaining a contradiction that $D_R(0)$ has a component that we will call E which does not contain 0. Within E we have $LG = 0$, $Lu_0 \leq 0$, and $u_0 < G$. On the other hand, it follows from [3] that E is a bounded set, and since $u_0 = G$ on ∂E , we contradict the weak maximum principle. ■

Lemma 2.6 (Density Result). *Assume $y_0 \in \partial D_r(x_0)$, and assume that c and C are the constants given in Theorem 2.2. Fix $h \in (0, 1/2)$. There exists a positive constant τ such that*

$$(2.7) \quad \frac{|B_{chr}(y_0) \cap D_r(x_0)|}{|B_{chr}(y_0)|} \geq \tau .$$

Note that τ has no dependence on x_0, y_0 , or r .

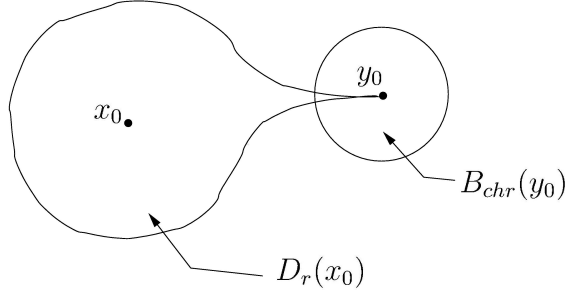


FIGURE 1. Not possible according to the lemma.

Proof. Without loss of generality we can rescale so that $r = 1$. Observe that Theorem 2.2 implies that x_0 belongs to the complement of $B_{ch}(y_0)$. It follows from the characterization of $D_1(x_0)$ as the noncontact set for an obstacle problem along with the nondegeneracy result of the second author and Hao (see Theorem 3.9 of [3]) that there exists a point z_0 at a distance of $ch/2$ to y_0 where the solution to the corresponding obstacle problem has grown by an amount $\sim h^2$. Next, by applying optimal regularity (see Theorem 3.2 of [3]) we can be sure that there is a ball with a radius bounded from below by a constant times h which is centered at z_0 which is not in the contact set. ■

Lemma 2.7 (Convergence of Minimizers). *For any $q > 0$, we let u_q minimize J_q within the set:*

$$(2.8) \quad K_{M,G} := \{u \in W^{1,2}(B_M) : u - G \in W_0^{1,2}(B_M), \text{ and } u \leq G \text{ a.e.}\}$$

where J_q is as given in Equation (2.6) above. Now fix $r > 0$. Then

$$(2.9) \quad u_s \rightharpoonup u_r \text{ in } W^{1,2}(B_M)$$

and

$$(2.10) \quad \lim_{s \rightarrow r} \|u_s - u_r\|_{C^\alpha(\overline{B_M})} = 0$$

for some $\alpha > 0$.

Proof. It is not hard to show that if s_m is a sequence of positive numbers converging to r , and if we let $u_m := u_{s_m}$, then the sequence $\{u_m\}$ is uniformly bounded in $W^{1,2}(B_M)$ and uniformly bounded in $C^\alpha(\overline{B_M})$. (See section 4 of [3] for details.) Thus, by using standard functional analysis we can be sure that there is a subsequence of s_m which we will denote by s_j such that we have

$$(2.11) \quad u_j \rightharpoonup \tilde{u} \text{ in } W^{1,2}(B_M) \text{ and } \lim_{j \rightarrow \infty} \|u_j - \tilde{u}\|_{C^\alpha(\overline{B_M})} = 0$$

for some function $\tilde{u} \in W^{1,2}(B_M) \cap C^\alpha(\overline{B_M})$. Since the original sequence $\{s_m\}$ was arbitrary, it remains only to show that $\tilde{u} = u_r$.

First note that for all of the u_m in our sequence, we have:

$$(2.12) \quad |J_r(u_m) - J_{s_m}(u_m)| \leq \int_{B_M} |2(s_m)^{-n} - 2r^{-n}| u_m \leq |2(s_m)^{-n} - 2r^{-n}| \tilde{C}$$

where \tilde{C} is the maximum of the L^1 norms of the u_m . Of course, as we let $m \rightarrow \infty$ the right hand side goes to zero. We know

$$\begin{aligned} J_r(u_r) &\leq J_r(\tilde{u}) && \text{because } u_r \text{ minimizes } J_r \\ &\leq \liminf_{j \rightarrow \infty} J_r(u_j) && \text{by weak lower semicontinuity} \\ &= \liminf_{j \rightarrow \infty} J_{s_j}(u_j) && \text{by using Equation (2.12).} \end{aligned}$$

Now we claim that

$$(2.13) \quad \liminf_{j \rightarrow \infty} J_{s_j}(u_j) \leq J_r(u_r)$$

which we can combine with the chain of inequalities in the previous paragraph along with uniqueness of minimizers to show that $\tilde{u} = u_r$. Suppose that this is not the case. Then there exists $s_k \rightarrow r$ and an $\epsilon > 0$ such that

$$(2.14) \quad J_{s_k}(u_k) \geq J_r(u_r) + \epsilon .$$

On the other hand, for sufficiently large k , by using Equation (2.12) again and then Equation (2.14) we have

$$J_{s_k}(u_r) \leq J_r(u_r) + \epsilon/2 \leq J_{s_k}(u_k) - \epsilon/2 < J_{s_k}(u_k)$$

which contradicts the fact that u_k is the minimizer of J_{s_k} . ■

Remark 2.8 (Statement for the w_R). Of course in the language of the height functions w_R , as long as K is compactly contained in the complement of $\{x_0\}$ we have

$$(2.15) \quad \lim_{r \rightarrow s} \|w_r - w_s\|_{C^\alpha(K)} = 0 .$$

Lemma 2.9 (Continuous Expansion). *Fix $x_0, y_0 \in \Omega$ and assume that there exists $0 < s < t$ so that y_0 is not contained in $D_s(x_0)$, and is compactly contained within $D_t(x_0)$. Then there exists a unique $r \in (s, t)$ such that $y_0 \in \partial D_r(x_0)$.*

Proof. We borrow some of the ideas used in the proof of the counter-example within [4]. Define the set of real numbers:

$$S := \{ t \in \mathbb{R} : y_0 \notin D_t(x_0) \} ,$$

and let r_0 be the supremum of S . Because the $D_r(x_0)$ are an increasing family of sets with respect to r , the set S is an interval. We claim that $y_0 \in \partial D_{r_0}(x_0)$. Assuming that $y_0 \notin \partial D_{r_0}(x_0)$, then there exists a $\rho > 0$ so that

$$(2.16) \quad \text{dist}(y_0, \partial D_{r_0}(x_0)) = \rho .$$

At this point there are two possible cases: In the first case $B_\rho(y_0) \subset D_{r_0}(x_0)$, and in the second case $B_\rho(y_0) \subset D_{r_0}(x_0)^c$.

Suppose first that $B_\rho(y_0) \subset D_{r_0}(x_0)$. In this case, we have $\Pi := \overline{B_{\rho/2}(y_0)} \subset D_{r_0}(x_0) = \{w_{r_0} > 0\}$, and so if

$$\gamma := \min_{\Pi} w_{r_0} ,$$

then $\gamma > 0$. By Lemma 2.7, there exists a sufficiently small $\delta > 0$ such that $|r - r_0| < \delta$ implies

$$(2.17) \quad \|w_r - w_{r_0}\|_{L^\infty(\Pi)} \leq \gamma/2 .$$

Then the triangle inequality implies $w_r \geq \gamma/2 > 0$ in all of $\Pi \subset D_{r_0}(x_0)$ which contradicts the definition of r_0 .

Next suppose that $B_\rho(y_0) \subset D_{r_0}(x_0)^c = \{w_{r_0} = 0\}$. Within $B_\rho(y_0)$ the function w_r satisfies the obstacle problem:

$$(2.18) \quad Lw_r = \chi_{\{w_r > 0\}} r^{-n}$$

and therefore w_r enjoys the quadratic nondegeneracy property. (See section 3 of [3].) Because of this nondegeneracy, as long as $r > r_0$, (and by using the definition of r_0), we can guarantee that there is a point within $\Pi := B_{\rho/2}(y_0)$ where w_r is greater than a constant $\gamma > 0$. On the other hand, by Lemma 2.7 again, there exists a sufficiently small $\delta > 0$ such that $|r - r_0| < \delta$ implies

$$(2.19) \quad \|w_r - w_{r_0}\|_{L^\infty(\Pi)} \leq \gamma/2.$$

Thus

$$0 < \gamma \leq \|w_r\|_{L^\infty(\Pi)} = \|w_r - w_{r_0}\|_{L^\infty(\Pi)} \leq \gamma/2$$

which gives us a contradiction for this case. Hence we must have $y_0 \in \partial D_{r_0}(x_0)$. ■

3. APPLICATIONS TO A BERNOULLI-TYPE FREE BOUNDARY PROBLEM

We turn now to applications of the mean value results to the following problem: Given $a^{ij}(x)$ as above, and boundary data, $\varphi \geq 0$ given on ∂B_1 , we consider minimizers of the functional:

$$J_a(u) := \int_{B_1} (a^{ij} D_i u D_j u + \chi_{\{u > 0\}})$$

which we gave above in Equation (1.6) for a general domain D . Now in the case where $a^{ij} \equiv \delta^{ij}$ the functional $J_a(u)$ simplifies to:

$$J(u) := \int_{B_1} (|\nabla u|^2 + \chi_{\{u > 0\}}).$$

Alt and Caffarelli considered local minimizers of this functional, and indeed this problem was used as a model problem within the text by Caffarelli and Salsa. We will say that u_0 is a local minimizer of J , if given any subdomain D_0 of B_1 the value of

$$J(u_0; D_0) := \int_{D_0} (|\nabla u_0|^2 + \chi_{\{u_0 > 0\}}),$$

is less than or equal to the value of $J(v; D_0)$ for any v which is equal to u_0 on ∂D_0 .

Some highlights of what is known about functions u_0 which locally minimize $J(u)$ in B_1 include the following:

Theorem 3.1 (Basic Facts for Minimizers of J). *Within any $D_0 \subset\subset B_1$ we have:*

(1) u_0 is Lipschitz.

(2) If $z_0 \in D_0 \cap \partial\{u_0 > 0\}$, then there is a constant $C > 0$ depending only on n and $\|u_0\|_{L^2(B_1)}$ such that

$$(3.1) \quad C^{-1}r \leq \sup_{B_r(z_0)} u_0 \leq Cr .$$

(3) With $z_0 \in D_0 \cap \partial\{u_0 > 0\}$ again, there is a universal $\theta > 0$ such that

$$(3.2) \quad \mathcal{L}^n(\{u_0 > 0\} \cap B_r(z_0)) \geq \theta r^n \quad \text{and} \quad \mathcal{L}^n(\{u_0 = 0\} \cap B_r(z_0)) \geq \theta r^n$$

where we use $\mathcal{L}^n(S)$ to denote the n -dimensional Lebesgue measure of S .

(4) Using $\mathcal{H}^\gamma(S)$ to denote the γ -dimensional Hausdorff measure of S , then given $D_0 \subset\subset B_1$ there is a universal C such that

$$(3.3) \quad \mathcal{H}^{n-1}(\partial\{u_0 > 0\} \cap D_0) \leq C.$$

(5) $|\nabla u_0| = 1$ in a suitable sense on almost all of the free boundary.

Everything in the theorem above was proven by Alt and Caffarelli. See [1, 5] for details.

More recently, dos Prazeres and Teixeira studied the local minimizers of the more general functional J_a where the a^{ij} which appear are assumed to be no more than bounded, symmetric, and uniformly elliptic. Now in this case, there is no hope of proving that minimizers are better than the Hölder regularity given by the famous result of De Giorgi and Nash. On the other hand dos Prazeres and Teixeira proved that functions u_0 which locally minimize $J_a(u)$ in B_1 satisfy the following:

Theorem 3.2 (Basic Facts for Minimizers of J_a). *Within any $D_0 \subset\subset B_1$ we have:*

(1) If $z_0 \in D_0 \cap \partial\{u_0 > 0\}$, then there is a constant $C > 0$ depending only on n, λ, Λ , and $\|u_0\|_{L^2(B_1)}$ such that

$$(3.4) \quad C^{-1}r \leq \sup_{B_r(z_0)} u_0 \leq Cr .$$

(2) With $z_0 \in D_0 \cap \partial\{u_0 > 0\}$ again, there is a universal $\theta > 0$ such that

$$(3.5) \quad \mathcal{L}^n(\{u_0 > 0\} \cap B_r(z_0)) \geq \theta r^n.$$

See [9, Theorem 1.1]. Also considered by dos Prazeres and Teixeira were a^{ij} satisfying what they called the “ K -Lip” property which do allow for Lipschitz estimates of the minimizers, but we never make this assumption. (For those details, see [9, Definition 3.3].) Of course, even without any further hypotheses, one can reasonably view Equation

(3.4) as saying that “at the free boundary” the solutions enjoy a Lipschitz-type behavior. On the other hand, for general a^{ij} one can easily construct a counter-example to the statement: “The one sided gradient exists at the free boundary” by choosing a^{ij} as in the paper by Blank and Teka, and then doing a blow up argument. Thus, it seems very difficult to get a successful analogue of the fifth statement in Theorem 3.1 above. It also seems difficult or impossible to prove Equation (3.3) in the general case, although as dos Prazeres and Teixeira observed, the free boundary is necessarily porous, and so if one is willing to weaken \mathcal{H}^{n-1} measure to $\mathcal{H}^{n-\zeta}$ measure for a ζ which is between 0 and 1, then one can assert the analogue [9]. From a certain point of view, the upshot is that the biggest gap between Theorem 3.1 and Theorem 3.2 that we can hope to close is the fact that Equation (3.5) is only giving half of what Equation (3.2) gave, and that leads to our first application.

3.1. Application 1: Positive Density of the Contact Set on the Free Boundary.

Theorem 3.3 (Positive Density of the Contact Set on the Free Boundary). *In the same setting as in Theorem 3.2 and with $z_0 \in D_0 \cap \partial\{u_0 > 0\}$ there exists a $\theta > 0$ depending only on n, λ, Λ , and $\|u_0\|_{L^2(B_1)}$ such that*

$$(3.6) \quad \mathcal{L}^n(\{u_0 = 0\} \cap B_r(z_0)) \geq \theta r^n.$$

Proof. Let v be a solution of the equation $Lu = 0$ in $B_r(x_0)$ with $v = u_0$ on $\partial B_r(x_0)$. Since x_0 is in the free boundary we know that u_0 and therefore v is positive on a nontrivial portion of $\partial B_r(x_0)$. Then, the strong maximum principle implies $v > 0$ in $B_r(x_0)$. Since u_0 is local minimizer we have,

$$\int_{B_r(x_0)} ((A(x)\nabla u_0) \cdot \nabla u_0 + \chi_{\{u_0 > 0\}}) \leq \int_{B_r(x_0)} ((A(x)\nabla v) \cdot \nabla v + \chi_{\{v > 0\}})$$

which gives,

$$\begin{aligned} \int_{B_r(x_0)} ((A(x)\nabla u_0) \cdot \nabla u_0 - (A(x)\nabla v) \cdot \nabla v) &\leq \int_{B_r(x_0)} \chi_{\{v > 0\}} - \int_{B_r(x_0)} \chi_{\{u_0 > 0\}} \\ &= |B_r(x_0)| - |\{u_0 > 0\} \cap B_r(x_0)| \\ &= |\{u_0 = 0\} \cap B_r(x_0)| \\ &= |\Omega_0^c \cap B_r(x_0)|. \end{aligned}$$

On the other hand we claim that,

$$\begin{aligned}
\int_{B_r(x_0)} \left((A(x)\nabla u_0) \cdot \nabla u_0 - (A(x)\nabla v) \cdot \nabla v \right) &= \int_{B_r(x_0)} \left(A(x)\nabla(u_0 - v) \right) \cdot \nabla(u_0 - v) \\
&\geq \lambda \int_{B_r(x_0)} |\nabla(u_0 - v)|^2 \\
&\geq \frac{C\lambda}{r^2} \int_{B_r(x_0)} |(u_0 - v)|^2.
\end{aligned}$$

Thus, if we grant the claim, then we obviously have

$$(3.7) \quad |\Omega_0^c \cap B_r(x_0)| \geq \frac{C\lambda}{r^2} \int_{B_r(x_0)} |(u_0 - v)|^2.$$

Turning to the proof of the claim we see immediately that the last two inequalities in the chain of inequalities above simply use uniform ellipticity and the Poincare inequality respectively. Thus our claim is proved if we show the first equality. So letting $\varphi := u_0 - v$ and observing that $\varphi \in W_0^{1,2}(B_r(x_0))$ we compute

$$\begin{aligned}
&\int_{B_r(x_0)} (A(x)\nabla u_0) \cdot \nabla u_0 - (A(x)\nabla v) \cdot \nabla v - (A(x)\nabla(u_0 - v)) \cdot \nabla(u_0 - v) \\
&= 2 \int_{B_r(x_0)} \left((A(x)\nabla u_0) \cdot \nabla v - (A(x)\nabla v) \cdot \nabla v \right) \\
&= 2 \int_{B_r(x_0)} (A(x)\nabla v) \cdot \nabla(u_0 - v) \\
&= 2 \int_{B_r(x_0)} (A(x)\nabla v) \cdot \nabla(\varphi) \\
&= 0
\end{aligned}$$

since $Lv = 0$ in $B_r(x_0)$. Thus, the claim is proved.

Now using the MVT for general divergence form operators we get,

$$\begin{aligned}
v(x_0) &= \frac{1}{|D_r(x_0)|} \int_{D_r(x_0)} v \\
&\geq \frac{1}{|B_{Cr}(x_0)|} \int_{B_{Cr}(x_0)} v \\
&= \frac{|B_{Cr}(x_0)|}{|B_{Cr}(x_0)|} \cdot \frac{1}{|B_{Cr}(x_0)|} \int_{B_{Cr}(x_0)} v \\
&\geq \tilde{C} \frac{1}{|B_{Cr}(x_0)|} \int_{B_{Cr}(x_0)} u_0 \\
&\geq \tilde{C}r
\end{aligned}$$

where in the final inequality we have used both the nondegeneracy and the optimal regularity of u_0 due to dos Prazeres and Teixeira [9]. Since v is L -harmonic and nonnegative, the Harnack inequality tells us that $v(y) \geq \tilde{C}r$ for all $y \in B_{r/2}(x_0)$. By the Lipschitz continuity of u_0 we see that $u_0(y) \leq c_1hr$ in $B_{hr}(x_0)$. By choosing h to be sufficiently small we get

$$v - u_0 \geq \hat{c}r \quad \text{in } B_{hr}(x_0) .$$

Therefore by using Equation (3.7) we get,

$$|\Omega_0^c \cap B_r(x_0)| \geq \frac{c\lambda}{r^2} \int_{B_r(x_0)} |(u_0 - v)|^2 \geq \frac{c\lambda}{r^2} \int_{B_{hr}(x_0)} (\hat{c}r)^2 \geq Cr^n .$$

■

By combining this last result with part (2) of Theorem 3.2 we get the following statement simply by definition.

Corollary 3.4 (Measure Theoretic Boundary). *Every point of the free boundary belongs to the measure theoretic boundary of the zero set and/or of the positivity set.*

Definitions and information about the measure theoretic boundary can be found in a variety of references on geometric measure theory including [10] and [12].

3.2. Application 2: A Nondegeneracy Lemma.

Although the previous application of the MVT gives us a new result, it does not make use of the new properties that we have shown. On the other hand, by making use of our lemmas in the second section, we can give a new proof of many of the results shown independently by dos Prazeres and Teixeira. Indeed, our method of proof follows the exposition of Caffarelli and Salsa's text almost exactly, and so we will state here only the proof of the key lemma that relies on our statements of the $D_r(x_0)$. This lemma is the analogue of Lemma 1.10 of [5].

Lemma 3.5 (Nondegeneracy Lemma). *Let Ω be an open set with $0 \in \partial\Omega$ and $w \geq 0$, $\|w\|_{C^{0,1}(B_2)} = \bar{\mathcal{K}}$, and $Lw = 0$ in $\Omega \cap B_2$. Suppose $x_0 \in \Omega \cap B_1$ and*

- (i) $w(x_0) = \sigma > 0$, and
- (ii) *in the region $\{w \geq \sigma/3\}$, we have $w(x) \sim \text{dist}(x, \partial\Omega)$.*

Then there exist positive constants η, β, γ , and σ_0 which all depend on n, λ, μ , and \bar{K} , such that as long as $\sigma \leq \sigma_0$, we have

$$(3.8) \quad \beta\sigma \geq \sup_{B_{\eta\sigma}(x_0)} w \geq (1 + \gamma)\sigma .$$

Proof. Define $\rho > 0$ by

$$(3.9) \quad \rho := \sup\{r \in \mathbb{R} : D_r(x_0) \subset \{w > \sigma/3\}\} ,$$

where $D_r(x_0)$ is the solid mean value set given in Theorem 2.2. Using Lemma 2.9 there exists a $y_0 \in \partial D_\rho(x_0)$ with $w(y_0) = \sigma/3$. By assumptions (i) and (ii) we know that $\rho \sim \sigma$. By the Lipschitz continuity of w , for a suitable $h > 0$, we have $w(x) \leq 2\sigma/3$ for all $x \in B_{h\rho}(y_0)$. Now by using Lemma 2.6 we know that $w \leq 2\sigma/3$ in a fixed proportion of $D_\rho(x_0)$. By the basic properties of the mean value sets $D_r(x_0)$, we have:

$$(3.10) \quad \sigma = w(x_0) = \int_{D_\rho(x_0)} w(y) dy ,$$

but since there is a fixed proportion of $D_\rho(x_0)$ where w is less than $2\sigma/3$ we must have a point in $D_\rho(x_0)$ which exceeds σ by some fixed amount. Since $D_\rho(x_0) \subset B_{C\rho}(x_0)$ with C as given in Theorem 2.2, and since as we observed above we have $\rho \sim \sigma$, we get the right hand side of Equation (3.8). The left hand side of Equation (3.8) follows trivially from Lipschitz continuity so we are done. \blacksquare

Iterating this lemma in the same fashion that Caffarelli and Salsa iterate their Lemma 1.10 leads to the key nondegeneracy theorem for solutions to this free boundary problem.

REFERENCES

- [1] H. W. Alt and L. A. Caffarelli. Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.*, 325:105–144, 1981.
- [2] Brian Benson, Ivan Blank, and Jeremy LeCrone. Mean value theorems for riemannian manifolds via the obstacle problem. *Submitted*, 2017.
- [3] Ivan Blank and Zheng Hao. The mean value theorem and basic properties of the obstacle problem for divergence form elliptic operators. *Comm. Anal. Geom.*, 23(1):129–158, 2015.
- [4] Ivan Blank and Kubrom Teka. The Caffarelli alternative in measure for the nondivergence form elliptic obstacle problem with principal coefficients in VMO. *Comm. Partial Differential Equations*, 39(2):321–353, 2014.
- [5] Luis Caffarelli and Sandro Salsa. *A geometric approach to free boundary problems*, volume 68 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.
- [6] Luis A. Caffarelli. *The obstacle problem*. Lezioni Fermiane. [Fermi Lectures]. Accademia Nazionale dei Lincei, Rome; Scuola Normale Superiore, Pisa, 1998.
- [7] Luis A. Caffarelli. The obstacle problem revisited. *J. Fourier Anal. Appl.*, 4(4-5):383–402, 1998.

- [8] Luis A. Caffarelli and Jean-Michel Roquejoffre. Uniform Hölder estimates in a class of elliptic systems and applications to singular limits in models for diffusion flames. *Arch. Ration. Mech. Anal.*, 183(3):457–487, 2007.
- [9] Disson dos Prazeres and Eduardo V. Teixeira. Cavity problems in discontinuous media. *Calc. Var. Partial Differential Equations*, 55(1):Art. 10, 15, 2016.
- [10] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.
- [11] W. Littman, G. Stampacchia, and H. F. Weinberger. Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa (3)*, 17:43–77, 1963.
- [12] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.